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NONSTATIONARY HEAT CONDUCTION OF THREE-DIMENSIONAL BODIES WITH UNEVEN INCLUSIONS

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A study is made of the nonstationary heat conduction of a three-dimensional body with an uneven inclusion mated with the matrix by an ideal thermal contact. Using thermal potential determined and studied on uneven surfaces, the problem has been reduced to an unconditionally solvable system of integral equations. The results have been illustrated with an example of sudden heating of a body with a circular conic inclusion.

Formulation of the Problem. Stationary temperature fields of two-dimensional and three-dimensional bodies with uneven (rough) inclusions have been studied in [1–4] based on the methods developed in [5–7]. We investigate the nonstationary heat conduction of a three-dimensional body with an uneven inclusion mated with the matrix by an ideal thermal contact. Let the unbounded body contain an uneven inclusion occupying a finite simply connected region V_1 and the inclusion surface S contain noninteracting smooth singular lines (sets of angular points) and conic points. The inclusion is mated with the matrix by an ideal thermal contact, and the composite is exposed to external nonstationary thermal actions. We find the temperature field of the composite.

We construct the solutions $T_j = T_j(x, y, z, t)$ of the nonstationary heat-conduction equation [8]

$$\nabla^2 T_j - a_j^2 \frac{\partial T_j}{\partial t} = f_j (x, y, z, t) \tag{1}$$

with initial conditions

$$T_j (x, y, z, t)\big|_{t=0} = 0 \tag{2}$$

and the following mating conditions on the surface S [8]:

at the points of smoothness

$$T_0^- (x, y, z, t) - T_1^+ (x, y, z, t) = 0 \ , \quad \lambda_0 \frac{\partial T_0^- (x, y, z, t)}{\partial n} - \lambda_1 \frac{\partial T_1^+ (x, y, z, t)}{\partial n} = 0 \ ; \tag{3}$$

at the singular points $M_0(x_0, y_0, z_0)$

$$\begin{gathered} \lim_{M \to M_0} [T_0^- (x, y, z, t) - T_1^+ (x, y, z, t)] = 0 \ , \\ \lim_{M \to M_0} \left[\lambda_0 \frac{\partial T_0^- (x, y, z, t)}{\partial n} - \lambda_1 \frac{\partial T_1^+ (x, y, z, t)}{\partial n} \right] = 0 \ . \end{gathered} \tag{4}$$

Here the subscripts j = 0 and 1 refer to quantities in the regions $V_0 = R_3 \backslash V_1$ (R_3 is the three-dimensional space) and V_1; n is the normal to the surface S (the normal is external in relation to the region V_1; the superscripts $\pm$ mean the

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boundary values of the quantities, as the point $M(x, y, z)$ approaches the surface S from the regions V_1 ("+" sign) or V_0 ("−" sign); $a_j^2 = \frac{1}{\kappa_j}$; $\kappa_j = \frac{\lambda_j}{c_j\rho_j}$; c_j and ρ_j are the specific heat and the density.

At the singular points M_0, the fulfillment of the heat-conduction equation is understood in the sense of realization of the equality [9]

$$\lim_{\Delta V \to M_0} \iiint_{\Delta V} [\nabla^2 T_j - a_j^2 \partial T_j/\partial t - f_j(x, y, z, t)]\, dv = 0 , \tag{5}$$

i.e., of the conservation of thermal balance upon the contraction of the elementary region ΔV to the singular point M_0.

Temperature Fields. We determine the behavior of the solution of the nonstationary heat-conduction equation (1) in the vicinity of the singular points of the mating surface S. The homogeneous equation corresponding to (1) is invariant relative to the transformation

$$x = Bx_1 , \quad y = By_1 , \quad z = Bz_1 , \quad t = B^2 t_1 , \quad x = Bx_1$$

(B is an arbitrary real number). With allowance for the linearity of mating conditions (3) and (4) and the homogeneity of initial condition (2), the solution of the problem is found from the functional equation

$$T_j(x, y, z, t) = A(B)\, T_j(Bx, By, Bz, B^2 t) , \tag{6}$$

where the quantity A is a function of B. Differentiating relation (6) with respect to B and introducing the notation $K = -\frac{B}{A}\frac{dA}{dB}$, we obtain the equation

$$(\mathrm{grad}\, T_j, \mathbf{r}) + 2t \frac{\partial T_j}{\partial t} = K T_j ,$$

whose solution determines the class of solutions of the heat-conduction equation

$$T_j = x^m \varphi_j \left(\frac{x}{y}, \frac{x}{z}, \frac{x}{\sqrt{t}} \right). \tag{7}$$

Let the interface S of the regions contain a smooth singular line. We write the representation (7) in the local coordinates ρ, θ, s connected with the singular line [7]. Satisfying relation (5) using the found T_j, we arrive at the governing equation for T_j, coincident with the corresponding equation for the harmonic function, and the known asymptotics [7]. In the case of a conic point, we assure ourselves, passing, in (7), to the curvilinear orthogonal coordinates ρ_1, θ_1, s_1 connected with this point [7], that the asymptotics of the solution of the homogeneous nonstationary heat-conduction equation and the asymptotics of the corresponding harmonic function coincide.

Thus, it has been established that the asymptotics of solutions of the nonstationary homogeneous heat-conduction equation corresponding to (1) and to mating conditions (3) and (4) in the vicinity of the singular points of the mating surface coincides with the asymptotics of harmonic functions with the same mating conditions [7]. Therefore, the distributions of the nonstationary temperature and the components of the heat-flux density as well as the formulas for computation of the intensity coefficients have the form given in [1]. It is noteworthy that an analogous circumstance holds for the wave Lamé equation [10].

The solution of Eq. (1) represents the sum of thermal potentials of the single and double layers [11] and the volume [12]. To solve the problem we must study the conditions of existence and the properties of the potentials in the case of uneven surfaces. If the density $\mu_2(M, t)$ of the thermal potential of the double layer $W(M, t)$ is a continuous function at the points of smoothness $M(x, y, z)$ of the surface S, is equal to zero at the singular points of the surface $M_0(x_0, y_0, z_0)$, and has the asymptotics

$$\mu_2 (M, t) = O(|MM_0|^{\alpha}), \quad \alpha \in (0, 1), \quad t \in (0, \infty), \tag{8}$$

in their vicinity, the thermal potential exists, satisfies the homogeneous nonstationary heat-conduction equation, and, at the points of smoothness M, we have the known representation for its boundary values

$$W^{\pm} (M, t) = \pm 2\pi\mu_2 (M, t) + W (M, t), \tag{9}$$

where $W(M, t)$ is the direct value of the potential and $W^{\pm}(M, t)$ are its boundary values, as the point of smoothness of the surface is approached along the external normal $\mathbf{n}$ from the region V_1 ("+" sign) or V_2 ("–" sign).

Indeed, let the surface S contain one smooth singular line. We surround this line with a tubular surface S_l of radius l. As a result, the surface S will be broken into the surfaces $S^{(1)}$ and $S^{(2)}$ lying inside the tubular surface and outside it. We write the thermal potential of the double layer as the sum

$$W (M, t) = \int_0^t d\tau \iint_{S^{(1)}} \mu_2 (N, \tau) K_2 (M, N, t, \tau)\, ds_N + \int_0^t d\tau \iint_{S^{(2)}} \mu_2 (N, \tau) K_2 (M, N, t, \tau)\, ds_N, \tag{10}$$

where the kernel is

$$K_2 (M, N, t, \tau) = \frac{\partial}{\partial n} K_1 (M, N, t, \tau),$$

$$K_1 (M, N, t, \tau) = \frac{a}{2\sqrt{\pi}\,(t-\tau)^{3/2}} \exp\left(-\frac{r^2 a^2}{4(t-\tau)}\right), \quad r = |MN|.$$

The second term of the right-hand side of equality (10) represents the thermal potential of the double layer on Lyapunov's open surface, and its boundary values at the points of smoothness $M \in S^{(2)}$ are as follows [11]:

$$\left[\int_0^t d\tau \iint_{S^{(2)}} \mu_2 (N, \tau) K_2 (M, N, t, \tau)\, ds_N\right]_M^{\pm} = \pm 2\pi\mu_2 (M, t) + \int_0^t d\tau \iint_{S^{(2)}} \mu_2 (N, \tau) K_2 (M, N, t, \tau)\, ds_N. \tag{11}$$

The first term in (10), which is computed at $M \in S^{(2)}$ as an improper integral with the use of the local coordinates ρ, s connected with the singular line [7] and with account for (8), is a continuous function in passage of the point M through the surface $S^{(2)}$. Therefore, equalities (10) and (11) yield the truth of the representation (9).

In the case of a conic singular point, the proof is constructed according to the scheme presented above with the use of the local coordinates connected with such a point [7]. The proof for a finite number of nonintersecting smooth singular lines and conic points not belonging to them is analogous. Direct substitution of the expression for the double-layer potential into the homogeneous equation corresponding to (1) shows that it is satisfied outside the surface S.

If the density $\mu_1(M, t)$ of the thermal potential of the single layer $V(M, t)$ is a continuous function at the points of smoothness of the surface S and, at the singular points M, tends to infinity with the asymptotics

$$\mu_1 (M, t) = O\left(\frac{1}{|MM_0|^{\alpha}}\right), \quad \alpha \in (0, 1), \quad t \in (0, \infty), \tag{12}$$

the thermal potential of the single layer exists, satisfies the homogeneous heat-conduction equation, and is a continuous function at the points of smoothness of the surface S. We have the known representation for the boundary values of the normal derivative of the potential

$$\left[\frac{\partial}{\partial n} V(M, t)\right]^{\pm} = \pm\, 2\pi\mu_1 (M, t) + \frac{\partial}{\partial n} V(M, t)\,, \tag{13}$$

where $\dfrac{\partial}{\partial n}$ $V(M, t)$ is the direct value of the normal derivative of the thermal potential of the single layer.

Indeed, let the surface contain a singular line. Analogously to the previous proof for the surface $S = S^{(1)} \cup S^{(2)}$, we have

$$V(M, t) = \int_0^t d\tau \iint_{S^{(1)}} \mu_1 (N, \tau)\, K_1 (M, N, t, \tau)\, ds_N + \int_0^t d\tau \iint_{S^{(2)}} \mu_1 (N, \tau)\, K_1 (M, N, t, \tau)\, ds_N\,. \tag{14}$$

The first integral on the right-hand side of (14) exists and represents the thermal potential of the single layer on Lyapunov's open surface [11], whereas the second integral, on passage to the variables ρ and s determined by the singular line [7] and with account for condition (12), is computed as the improper one. As the point of smoothness passes through the surface $S^{(2)}$, the second term of the right-hand side of (14) is continuous as the potential of the single layer on Lyapunov's open surface [11] and the first term at $M \notin S^{(1)}$ is a continuous function. Therefore, the potential $V(M, t)$ represents a continuous function upon passage of the point of smoothness through the surface S.

The normal derivative of equality (14) with account for the known boundary values of the normal derivative of the thermal potential of the single layer on Lyapunov's open surface [11] and for the continuity of the normal derivative of the first term of the right-hand side of (14) at $M \notin S^{(1)}$ has boundary values determined by formula (13).

The proof in the case of a conic point and a finite number of smooth nonintersecting singular lines and conic points not belonging to them is analogous. Substitution of the expression for the single-layer thermal potential into the homogeneous nonstationary heat-conduction equation shows that it is satisfied.

An analog of the Lyapunov–Tauber theorem holds, namely: if the density of the double-layer thermal potential satisfies conditions (8) and the limiting value of the normal derivative of the thermal potential exists at the point of smoothness on one side of the mating surface S, it also exists on the other side and the equality

$$\left[\frac{\partial}{\partial n} W(M, t)\right]^{+} = \left[\frac{\partial}{\partial n} W(M, t)\right]^{-}\,. \tag{15}$$

holds. Indeed, we consider the case where a singular point is present on the mating surface S. Separating, as has been described above, the surfaces $S^{(1)}$ and $S^{(2)}$, based on equality (10) we obtain

$$\frac{\partial}{\partial n} W(M, t) = \frac{\partial}{\partial n}\left[\int_0^t d\tau \iint_{S^{(1)}} \mu_1 (N, \tau)\, K_1 (M, N, t, \tau)\, ds_N\right] + \frac{\partial}{\partial n}\left[\int_0^t d\tau \iint_{S^{(2)}} \mu_1 (N, \tau)\, K_1 (M, N, t, \tau)\, ds_N\right]. \tag{16}$$

The second term on the right-hand side of relation (16) has the character of a normal derivative of the double-layer thermal potential on Lyapunov's open surface ([13], p. 91), i.e., if the limiting value of the term exists on one side of the surface $S^{(2)}$, the limiting value equal to it exists on the other side. The first integral on the right-hand side of (16) is computed with the use of the variables ρ and s [7] as the singular integral which is a continuous function in passage of the point of smoothness through the surface $S^{(2)}$; this yields the truth of the theorem. The proof in the case of a conic point and a finite number of nonintersecting singular points and conic points not belonging to them is analogous.

If the singular characteristic equations determined by conic points [7] have positive roots less than unity at these points, the quantity $m = \dfrac{\pi}{2\pi - \omega} < 1$ $(0 \le \omega < \pi)$, where ω is the apex angle of the mating surface in the plane normal to the singular line from the external normal, and the right-hand sides of Eqs. (1) $f_j(x, y, z, t)$ $(j = \overline{0, 1})$ are bounded in the corresponding regions V_j, then the problem of nonstationary heat-conduction (1)–(5) is unconditionally solvable.

We represent the temperature fields as the sum of the singular and regular components:

$$T_j(M,t) = T_{j1}(M,t) + T_{j2}(M,t) \quad (j = \overline{0,1}) . \tag{17}$$

We write the singular components in the form of the sum of the thermal potentials of the single and double layers:

$$T_{j1}(M,t) = V_{j1}(M,t) + W_{j1}(M,t) \tag{18}$$

with densities $\mu_{j1}(N)$ and $\mu_{j2}(N)$, respectively, which, in view of the presence of the positive roots of the singular characteristic equations, which are less than unity, are constructed just as those in [7]. As a result, we obtain

$$V_{j1}(M,t) = \frac{1}{a_j}\iint_S \frac{1}{r}\,\mathrm{erfc}\left(\frac{ra_j}{2\sqrt{t}}\right)\mu_{j1}(N)\,ds_N , \tag{19}$$

$$W_{j1}(M,t) = -\frac{16}{\sqrt{\pi}}\iint_S \frac{1}{r^2}\frac{\partial r}{\partial n_N}\left[\frac{ra_j}{2\sqrt{t}}\exp\left(-\frac{r^2a_j^2}{4t}\right) + \frac{\sqrt{\pi}}{2}\,\mathrm{erf}\left(\frac{ra_j}{2\sqrt{t}}\right)\right]\mu_{j2}(N)\,ds_N , \tag{20}$$

where erf and erfc are, respectively, the probability integral and the additional probability integral [14].

We represent the regular terms of the solutions by the sum of the thermal potentials of the single and double layers and the volume

$$T_{j2}(M,t) = a_{j2}(M,t) + V_{j2}(M,t) + W_{j2}(M,t) + U_{j2}(M,t) , \tag{21}$$

where $a_{j2}(M, t)$ are the known functions determined by the form of external thermal actions and satisfying the homogeneous heat-conduction equation; here, $a_{02}(M, t) \to 0$ as $M \to \infty$:

$$V_{j2}(M,t) = \int_0^t d\tau \iint_S \mu_{j1}^{(2)}(N,\tau)\,K_{j1}(M,N,t,\tau)\,ds_N ,$$

$$W_{j2}(M,t) = \int_0^t d\tau \iint_S \mu_{j2}^{(2)}(N,\tau)\,K_{j2}(M,N,t,\tau)\,ds_N ,$$

$$W_{j2}(M,t) = -\int_0^t f_j(N,\tau)\,K_{j1}(M,N,t,\tau)\,ds_N ,$$

$\mu_{j1}^{(2)}(N, \tau)$ and $\mu_{j2}^{(2)}(N, \tau)$ are the unknown densities. It has been allowed for in (21) that the thermal potential of the volume with a density $f(N, \tau)$ satisfies the inhomogeneous heat-conduction equation (1) ([12, p. 260]).

Satisfying the mating conditions at the points of smoothness (3) with (17)–(21) and using formulas (9), (13), and (15), we arrive at the system of integral equations in which we set, in view of the arbitrariness of the densities,

$$\mu_{01}^{(2)}(N,\tau) = \mu_{11}^{(2)}(N,\tau) , \quad \mu_{02}^{(2)}(N,\tau) = \mu_{12}^{(2)}(N,\tau) . \tag{22}$$

As a result, we obtain the system of integral equations

$$- 4\pi\mu_{02}^{(2)} (M, t) + \int_0^t d\tau \iint_S \mu_{01}^{(2)} (N, \tau) [K_{01} (M, N, t, \tau) - K_{11} (M, N, t, \tau)] \, ds_N +$$

$$+ \int_0^t d\tau \iint_S \mu_{02}^{(2)} (N, \tau) [K_{02} (M, N, t, \tau) - K_{12} (M, N, t, \tau)] \, ds_N = g (M, t) \, ,$$

$$- 2\pi (\lambda_0 + \lambda_1) \, \mu_{01}^{(2)} (M, t) + \int_0^t d\tau \iint_S \mu_{01}^{(2)} (N, \tau) \left[\lambda_0 \frac{\partial}{\partial n} K_{01} (M, N, t, \tau) - \lambda_1 \frac{\partial}{\partial n} K_{11} (M, N, t, \tau)\right] ds_N + \quad (23)$$

$$+ \int_0^t d\tau \iint_S \mu_{02}^{(2)} (N, \tau) \left[\lambda_0 \frac{\partial}{\partial n} K_{02} (M, N, t, \tau) - \lambda_1 \frac{\partial}{\partial n} K_{12} (M, N, t, \tau)\right] ds_N = h (M, t) \, ,$$

where

$$g (M, t) = - \bar{T}_{01} (M, t) - \bar{a}_{02} (M, t) - U_0 (M, t) + \overset{+}{T}_{11} (M, t) + \overset{+}{a}_{12} (M, t) + U_1 (M, t) \, ;$$

$$h (M, t) = - \lambda_0 \frac{\partial}{\partial n} [\bar{T}_{01} (M, t) + \bar{a}_{02} (M, t) + U_0 (M, t)] + \lambda_1 \frac{\partial}{\partial n} [\overset{+}{T}_{11} (M, t) + \overset{+}{a}_{12} (M, t) + U (M, t)] \, .$$

The right-hand sides of the integral equations (23) are continuous functions on S due to the realization of mating conditions at the singular points (4), and their kernels are continuous at the points of smoothness of the mating surface S. Therefore, the system of equations (23) has the character of Volterra integral equations in the variable t and the character of Fredholm equations in Cartesian coordinates [15, 16]. Such a system is unconditionally solvable and has the analytical solution obtained, for example, by the method of successive approximations ([16, p. 485]).

To completely solve the problem we must add the condition of thermal balance in the regions containing singularities of the interface of media:

$$\iint_{S_0} (\text{grad}\, T, \mathbf{n}) \, ds = 0 \, , \quad (24)$$

where the surface S_0, in the case of a conic point, is a sphere with its center at this point and, in the case of a singular line, is a closed surface covering the line mentioned.

As an example, we study the heat conduction of a body containing a circular conic inclusion bounded by the surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{(z + c)^2}{c^2} = 0 \, ,$$

$$z = 0 \quad (25)$$

in sudden heating of the composite by the quantity ΔT (Fig. 1). First we solve the auxiliary problem in a three-dimensional space

$$\nabla^2 T - a_0^2 \frac{\partial T}{\partial t} = 0 \, , \quad a_0^2 = \frac{1}{\kappa_0} \quad (26)$$

with the initial condition

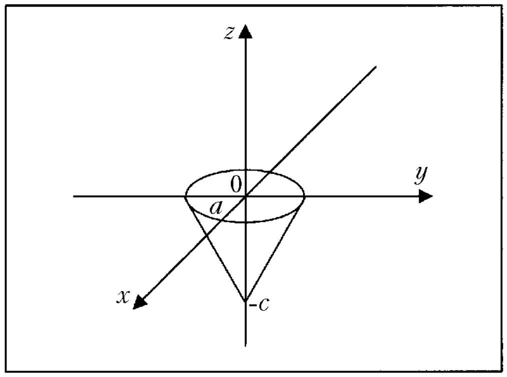

Fig. 1. Geometric diagram of a circular conic inclusion.

$$T\,(x, y, z, t)\big|_{t=0} = \Delta T\,. \tag{27}$$

Applying the Laplace transformation to (26) with account for (27), we obtain the equation

$$\nabla^2\bar{T} - k^2\bar{T} = -\frac{1}{\kappa_0}\,\Delta T\,, \quad k = \sqrt{\frac{p}{\kappa_0}}\,, \tag{28}$$

whose solution has the form [9]

$$\bar{T} = \frac{\Delta T}{4\pi\kappa_0}\iiint\limits_V R^{-1}\exp\,(-\,kR)\,d\xi d\eta d\chi\,, \quad R = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\chi)^2}\,. \tag{29}$$

Evaluating the integral (29) and recovering the original function, we find the explicit expression for the auxiliary temperature field

$$T\,(r, t) = \Delta T\left[\frac{r}{\sqrt{\pi\kappa_0 t}}\exp\left(-\frac{r^2}{4\kappa_0 t}\right) - \mathrm{H}\,(t) + \mathrm{erfc}\left(\frac{r}{2\sqrt{\kappa_0 t}}\right)\right], \tag{30}$$

where $\mathrm{H}(t)$ is the Heaviside function.

We represent the temperature fields in the form (17)–(20), where the singular components have the form (18), and construct the densities according to the distributions (given in [1]) of the components of the heat-flux density and the temperature. In accordance with the formulas of the singular components [7], with the use of the notation adopted in [7], and with account for the equations of the circular-cone surfaces S_1 and S_2 ($S_0 \equiv S_2$), we have

$$R_0\,(M) = R_1\,(M) = (z^2 + 0.25\,(x^2 + y^2 - a^2\,(z + c)\,c^{-1})^2\,(x^2 + y^2)^{-1})^{1/2}\,, \tag{31}$$

$$U\,(M) = \arcsin\,(z/R_{\mathrm{c}}\,(M))\,, \quad R_{\mathrm{c}}\,(M) = (x^2 + y^2 + (z + c)^2)^{1/2}\,,$$

$$U_{\mathrm{c}}\,(M) = \arcsin\,(\sin\theta_0\,(x^2 + y^2)^{1/2} + (\cos\theta_0 - c)\,(z + c))\,R_{\mathrm{c}}^{-1}\,(M)\,, \tag{32}$$

where $\theta_0 = \arctan\,(a/c)$.

The value of the index of singularity of the heat-flux density in the vicinity of the singular line has the form [1]

$$m = \frac{\pi}{2\pi - \omega}\,.$$

Here ω is the slope of the generator of the cone to the base. The singular characteristic equation determining the order of singularity of the heat-flux density in the vicinity of the vertex of the circular cone has the form [1, 2]

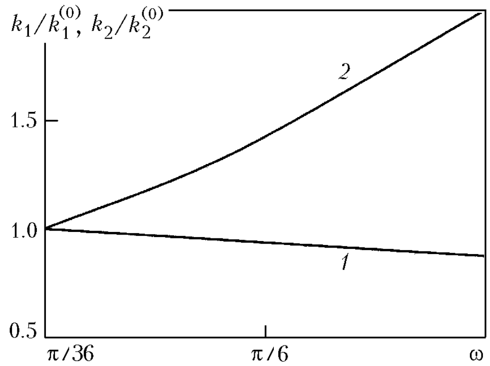

Fig. 2. Relative intensity factors vs. slope ω of the generator to the base of the cone at the instant of time determined by the equality $a/\sqrt{\kappa_0 t} = 0.2$: 1) $k_1/k_1^{(0)}$; 2) $k_2/k_2^{(0)}$.

$$P_{m_c}(\cos\theta_4) = 0\ ,$$

where P_{m_c} is the Legendre function; $\theta_4 = \theta_1 + \beta$; $\tan\beta = c/a$; the roots of this equation are tabulated [17].

The external nonstationary thermal action is represented according to formula (30): $a_{02}(x, y, z, t) = T(r, t)$. Substituting (18)–(21), (31), and (32) into boundary conditions (3), we arrive at the system of integral equations (23). In constructing the solution in iterative form, we assumed that $\gamma = \dfrac{\lambda_0}{\lambda_1} = 0.8$ (composite of the carbon steel–nickel type [18]) and took five iterations, which gave an error of 4.5%.

In view of the geometric (circular cone) and thermal symmetry of the problem (sudden heating by ΔT), we assume that the unknown coefficients C_{01}, C_{02}, and C_1 of the densities of the singular components of the temperature fields are constant. When thermal-balance condition (24) is fulfilled at the conic point and on the singular line we obtain the system of linear algebraic equations

$$A_1 C_{01} C_1 + B_1 C_{02} C_1 = D_1\ ,\quad A_2 C_{01} C_1 + B_2 C_{02} C_1 = D_2\ ,$$

where the quantities $A_q = A_q(a, c, \Delta T, t)$, $B_q = B_q(a, c, \Delta T, t)$, and $D_q = D_q(a, c, \Delta T_0, t)$ $(q = \overline{1, 2})$ depend on the geometry of the inclusion, the time, and the heat load and are not given because of their cumbersomeness:

Formula (24) yields the explicit expression for the intensity factor of the heat-flux density in the vertex of the conic inclusion:

$$k_0 = C_1\,(C_{01} + C_{02})\ . \tag{33}$$

Relations for the intensity factors of the heat-flux density in the vicinity of the singular line are given by the formulas [1]

$$k_1 = \lambda_0\,(\gamma - 1)\,(C_{01}\cos m\theta_1 + C_{02}\sin m\theta_1)/(\gamma\sqrt{2})\ ,$$
$$k_2 = \lambda_0\,(1 - \gamma)\,(C_{01}\sin m\theta_1 - C_{02}\cos m\theta_1)/\sqrt{2}\ , \tag{34}$$

where we must replace C_{01} and C_{02} by $C_{01}C_1$ and $C_{02}C_1$ respectively.

At values of t satisfying the inequality $r/\sqrt{\kappa_0 t} \le 0.02$, as follows from formula (30), we have $T(r, t) \approx 0$, i.e., the external thermal action is absent, which gives zero values of the coefficients of intensity of the heat flux according to formulas (33) and (34).

The dependence of the relative intensity factors $k_p/k_p^{(0)}$ $(p = \overline{1, 2})$ for the instant of time determined by the equality $a/\sqrt{\kappa_0 t} = 0.2$ on the value of the slope ω of the generator to the base is shown in Fig. 2. The quantities $k_1^{(0)} = \lambda_0((\gamma - 1)/(\gamma/2))C_{02}C_1$ and $k_2^{(0)} = \lambda_0(1 - \gamma)C_{01}C_1/\sqrt{2}$ are the factors of intensity of the heat-flux density in the vicinity of the singular line for the slope of the generator of the cone to the base, equal to $\pi/36$.

Basic Results and Conclusions. We have obtained a closed solution of the problem of nonstationary heat-conduction of a body with an uneven inclusion. It has been established that the distribution of the nonstationary temperature field and the components of the vector of the heat-flux density in the vicinity of the singularities of the interface of the media is identical to the corresponding distribution under stationary thermal actions.

The given example shows that, just as in the stationary case, the intensity of the heat-flux density in the vicinity of the singularities of the interfaces of media is characterized by both the order of singularity and the factors of intensity of the heat-flux density. Of primary importance is the order of singularity, and the intensity factors are a significant characteristic for one and the same order.

NOTATION

a and $-c$, radius of the base and z coordinate of the vertex of the cone; k_1 and k_2, intensity factors of the heat-flux density; l, radius of the tubular surface; p, complex number in the Laplace transform; T, temperature; t, time; V_0 and V_1, regions occupied by the body and the inclusion; x, y, z, Cartesian coordinates of the point M; $\gamma = \lambda_0/\lambda_1$; κ, thermal diffusivity; λ, thermal conductivity; μ_1 and μ_2, densities of the thermal potential of the single and double layers; ρ, θ, s, local coordinates connected with the singular line; ρ_1, θ_1, s_1, local coordinates connected with the conic point. Subscripts: $j = 0$ and 1 refer to the regions V_0 and V_1 respectively.

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